Presentation

Friday, July 17, 2020 11:14 AM

Stanley Lemma 3.1 :

$$X_P := \sum_{\substack{K \\ \text{"proper"}}} x_{K(v_1)} x_{K(v_2)} \cdots x_{K(v_d)} = \sum_{\alpha \in \mathcal{L}(P,w)} Q_{D(\alpha)}$$

Quasi-Symmetric Functions:

Definition: Given a power series $F(x_1, x_2, \ldots)$, $[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}] F(x)$ is the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ in $F(x)$.

Ex: $2e_2 = P_1^2 - P_2 = (x_1 + x_2 + \ldots)(x_1 + x_2 + \ldots) - (x_1^2 + x_2^2 + \ldots)$

So if $F(x) = 2e_2$, $[x_1 x_2]F(x) = 2$, $[x_k^2]F(x) = 0 \ \forall k \in \mathbb{N}$.

Definition: A power series $F(x)$ is quasi-symmetric if $[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}] F(x) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}] F(x)$ where the i_m & j_m sequences are strictly increasing.

Examples/Unexamples

1. $f(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + x_2^2 x_3 x_4$ ✓
2. $f(x_1, \ldots) = 1$ ✓
3. $f(x_1, x_2, x_3) = 2x_1 x_2 x_3 - 2x_1^2 x_3$ ✗
4. T/F Why? If f is symmetric, then f is q.s. ✓

$\mathcal{Q}_d$ & $Q_{S,d}$:

Definition: $\mathcal{Q}_d$ is the set of degree d homogeneous q.s. functions.

$\mathcal{Q}_d$ forms a vector space (over say $\mathbb{Q}$).

So $\mathcal{Q}_d$ has a basis $Q_{S,d}$ known as the fundamental basis.

$$Q_{S,d} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ if } j \in S \subseteq [d-1]}} x_{i_1} x_{i_2} \cdots x_{i_d}$$

Ex. $Q_{[d-1],d} = \sum_{i_1 < i_2 < \cdots < i_d} x_{i_1} x_{i_2} \cdots x_{i_d} = e_d$

$Q_{\emptyset,d} = \sum_{i_1 \le i_2 \le \cdots \le i_d} x_{i_1} x_{i_2} \cdots x_{i_d} = h_d$

Definition: A poset P is a set of elements with a binary relation $\le$ s.t. if $a, b, c \in P$

(1) $a \le a$

(2) $a \le b,\ b \le a \Rightarrow a = b$

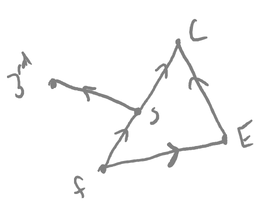
(3) $a \le b,\ b \le c \Rightarrow a \le c$

Very grounded example used for the rest of the presentation:

Let G be the poset with elements

Eric, Caitlin, fruit, fruit snacks, 3^{rd} amendment
E C f s 3^{rd}

$E < C$, $f < E$, $f < S$, $s < C$, $s < 3^{rd}$



$f < s$
$s < 3^{rd}$

Presentation Pg 2

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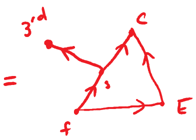
Linear extension: A linear extension of a poset P, is a bijection $\alpha : P \to [d]$ s.t. if $a, b \in P$ & $a < b$, then $\alpha(a) < \alpha(b)$.

Ex. $\alpha_1 = \begin{pmatrix} 3^{rd} & f & s & c & E \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}$

$G =$ 

$\alpha_2 = \begin{pmatrix} 3^{rd} & f & s & c & E \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$

An order reversing linear extension of P, ω, is a linear extension but with if $a < b$ for $a, b \in P$, then $\omega(a) > \omega(b)$.

Ex. $\omega = \begin{pmatrix} 3^{rd} & f & s & c & E \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$

Since $\alpha_i, \omega : P \to [d]$ (bijectively), $\omega \circ \alpha_i^{-1}$ is a permutation of $[d]$

Ex. Recall $\alpha_1 = \begin{pmatrix} 3^{rd} & f & s & c & E \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}$ then

$\omega \circ \alpha_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$

Definition: The descent set of α, $D(\alpha)$, is $\{i : a_i > a_{i+1}\}$ for a sequence $\{a_i\}$.

Ex: $D(\omega \circ \alpha_1^{-1}) \equiv D(\alpha) = D(52413) = \{1, \cancel{2}, 3, \cancel{4}\} = \{1, 3\}$

Definition: $\mathcal{L}(P, \omega) = \{\alpha : \alpha$ is a linear extension of $P\}$

So we understand $\sum_{\alpha \in \mathcal{L}(P,\omega)} Q_{D(\alpha)}$.

1. Given ω
2. Find α
3. Find $D(\alpha)$
4. Write as sum $Q_{D(\alpha)}$

$X_p := \sum_{K \text{ "proper"}} X_{K(v_1)} X_{K(v_2)} \cdots X_{K(v_d)}$ where K proper means if $a, b \in P$ & $a < b$, then $K(a) < K(b)$.

Example: $X_G = \sum_{K} x_{K(3^{rd})} x_{K(f)} x_{K(s)} x_{K(c)} x_{K(E)}$

s.t. $K(f) < K(s)$, $K(s) < K(c)$, $K(s) < K(3^{rd})$, $K(f) < K(E)$, $K(E) < K(c)$

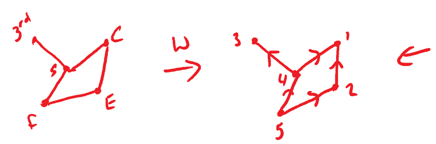
Notice that it is possible to get $x_{i_1} x_{i_2}^2 x_{i_3}^2$ by $K(f) < (K(E) = K(s)) < (K(3^{rd}) = K(c))$.

So colorings which look the same yield all possible $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ terms.
So maybe there's a nice connection between colorings & $[x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}] X_P$ & $Q_{s,d}$. (Hint: I chose $x_{i_1} x_{i_2}^2 x_{i_3}^2$ because it looks like $x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5}$ where if $k \in D(\alpha_1) = \{1, 3\}$, $j_k < j_{k+1}$ & if $k \notin D(\alpha_1) \setminus \{5\}$, $j_k = j_{k+1}$).

Example, find X_G using $\sum_{\alpha \in \mathcal{L}(P,\omega)} Q_{D(\alpha)}$

Exercise: Convince Yourself $\sum_{\alpha \in \mathcal{L}(G,\omega)} Q_{D(\alpha)} = \sum_{K \text{ "proper"}} x_{K(3^{rd})} x_{K(f)} x_{K(s)} x_{K(c)} x_{K(E)}$

Recall $G =$



1. Given ω
2. Find α
3. Find $D(\alpha)$
4. Write as $\sum$

$\omega \circ \alpha' = \{54321, 54231, 54213, 52413, 52431\}$

$D(\omega \circ \alpha_i^{-1}) = \{\{1,2,3,4\}, \{1,2,4\}, \{1,2,3\}, \{1,3\}, \{1,3,4\}\}$

$$X_G = Q_{\{1,2,3,4\}} + Q_{\{1,2,4\}} + Q_{\{1,2,3\}} + Q_{\{1,3\}} + Q_{\{1,3,4\}}$$

All terms will give $x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5}$ where $i_m < i_{m+1}$
So there's $5 x_1 x_2 x_3 x_4 x_5$